

WEAK COMPACTNESS AND FIXED POINT PROPERTY FOR AFFINE BI-LIPSCHITZ MAPS

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ABSTRACT. In this paper we show that if (y_n) is a seminormalized sequence in a Banach space which does not have any weakly convergent subsequence, then it contains a wide-(s) subsequence (x_n) which admits an equivalent convex basic sequence. This fact is used to characterize weak-compactness of bounded, closed convex sets in terms of the generic fixed point property (\mathcal{G} -FPP) for the class of affine bi-Lipschitz maps. We also introduce a relaxation of this notion (\mathcal{WG} -FPP) and observe that a closed convex bounded subset of a Banach space is weakly compact iff it has the \mathcal{WG} -FPP for affine 1-Lipschitz maps. Related results are also proved. For example, a complete convex bounded subset C of a Hlcs X is weakly compact iff it has the \mathcal{G} -FPP for the class of affine continuous maps $f: C \rightarrow X$ with weak-approximate fixed point nets.

1. INTRODUCTION

In a manifold of problems emerging in many branches of functional analysis, a great challenge is characterizing underlying topological phenomena. In particular, trends towards describing compactness have so far been of great interest. The *raison d'être* of this work comes from various related problems in Metric Fixed Point Theory. We will focus, here, on whether compactness can be interpreted by the fixed point property (FPP). Recall a topological space C is said to have the FPP for a class \mathcal{M} of maps if every $f \in \mathcal{M}$ with $f(C) \subset C$ has a fixed point. Topological aspects related to this problem have been carried out in diverse works, cf. [20, 11, 22, 5, 3, 4] and references therein. In the purely metric context, the problem is often subjected to structural considerations. This has been a keynote in many works on the weak-compact characterization of the FPP for affine nonexpansive (i.e. 1-Lipschitz) mappings. In [21] Lennard and Nezir proved for example that if a Banach space contains an asymptotically isometric c_0 -summing basic sequence (x_n) then $C = \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$ fails the FPP for affine nonexpansive mappings.

An interesting relaxation of the FPP is the generic-FPP (\mathcal{G} -FPP), a notion that was first introduced in [4]. Given a topological vector space (tvs, in short) X and a convex set $M \subset X$ denote by $\mathcal{B}(M)$ the family of all nonempty bounded, closed convex subsets of M .

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Definition 1.1 ([4]). A nonempty set $C \in \mathcal{B}(X)$ is said to have the \mathcal{G} -FPP for a class \mathcal{M} of mappings if whenever $K \in \mathcal{B}(C)$ then every mapping $f \in \mathcal{M}$, leaving invariant K , has a fixed point in K .

The authors Dowling, Lennard and Turett [7, 8] proved that when X is either c_0 , $L_1(0, 1)$ or ℓ_1 , sets in $\mathcal{B}(X)$ are weakly compact if and only if they have the \mathcal{G} -FPP for affine nonexpansive maps. In 2004 Benavides, Japón-Pineda and Prus proved the following.

Theorem 1.2 ([4]). *Let X be a Banach space and $C \in \mathcal{B}(X)$. Then C is weakly compact if and only if C has the \mathcal{G} -FPP for affine continuous maps. Furthermore,*

- (i) *if X is either c_0 (equipped with the supremum norm) or J_p (the James space), then these conditions are equivalent to the following: C has \mathcal{G} -FPP for affine uniformly Lipschitzian maps, and*
- (ii) *if X is an L -embedded Banach space, then C is weakly compact if and only if it has the \mathcal{G} -FPP for affine nonexpansive mappings.*

In 2008 P.-K. Lin [23] showed that if ℓ_1 is equipped with the equivalent norm

$$\|x\|_{\mathcal{L}} = \sup_{k \in \mathbb{N}} \frac{8^k}{1 + 8^k} \sum_{n=k}^{\infty} |x(n)| \quad \text{for } x = (x(n))_{n=1}^{\infty} \in \ell_1,$$

then every $C \in \mathcal{B}((\ell_1, \|\cdot\|_{\mathcal{L}}))$ has the FPP for nonexpansive maps. Consequently, since $(\ell_1, \|\cdot\|_{\mathcal{L}})$ is nonreflexive, the unit ball $B_{(\ell_1, \|\cdot\|_{\mathcal{L}})}$ fails to be weakly compact. But it turns out that it has the \mathcal{G} -FPP for affine nonexpansive maps. Another interesting result from the recent literature is due to T. Gallagher, C. Lennard and R. Popescu [12]. They exhibited a non-weakly compact set $C_0 \in \mathcal{B}((c, \|\cdot\|_{\infty}))$ with the FPP for nonexpansive mappings, where c is the Banach space of convergent scalar sequences. Nevertheless it is an open question as to whether or not on every non-weakly compact set $C \in \mathcal{B}(c_0)$ there exists an affine nonexpansive mapping f that is fixed point free.

With all these facts in mind it is natural to expect weak-compactness also to describe \mathcal{G} -FPP for affine uniformly Lipschitz maps in arbitrary Banach spaces. A technical difficulty is, however, how to get wide-(s) sequences which dominate all of its subsequences; that is, wide-(s) sequences (x_n) such that for some constant $L > 0$ and every $(n_i) \in [\mathbb{N}]$ with $n_i > i$ for all $i \in \mathbb{N}$,

$$\left\| \sum_{i=1}^n a_i x_{n_i} \right\| \leq L \left\| \sum_{i=1}^n a_i x_i \right\|$$

for all $n \in \mathbb{N}$ and all choice of scalars $(a_i)_{i=1}^n$. In such circumstances one can establish the failure of the \mathcal{G} -FPP in C for affine uniformly-Lipschitz maps.

It turns out that this estimate requirement brings out some unconditionality character. Indeed, unconditional basis as well as quasisubsymmetric basis (in sense of [1, Corollary 2.7]) are examples of basic sequences with such a property. So, it might be not easy to get it for free since unconditional basic sequences need not exist [13].

Another difficulty is that not all non-reflexive space X contains asymptotically isometric copies of ℓ_1 , a very useful device in this context. This was noted by Dowling, Johnson, Lennard and Turett in [6]. In contrast, its availability is higher when special structures are working out, see, e.g. [7, Theorem 1], [4, Theorem 4.2], [21] and [27, Proposition 2.5.14].

The main purpose of this paper is to solve the above problem for the class of affine bi-Lipschitz maps and explore a relaxation of the \mathcal{G} -FPP which seems to be reasonable for the context of 1-Lipschitz maps, in a most generalized sense of the word.

2. STRUCTURE OF SEMINORMALIZED SEQUENCES AND \mathcal{G} -FPP

The purpose of this section is to characterize weak-compactness in Banach spaces in terms of the \mathcal{G} -FPP for affine bi-Lipschitz maps. To be able to do that we first need to shed some light on the inherent structure of semi-normalized sequences. In what follows X will denote a Banach space with norm $\|\cdot\|$.

Definition 2.1. A sequence (x_n) in X is called *semi-normalized* if

$$0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty.$$

If (x_n) is basic then $[x_n]$ and \mathcal{K} stand respectively for its closed linear span and basic constant. In this case, we will denote by P_n and R_n the natural projections given by $P_n x = \sum_{i=1}^n x_i^*(x) x_i$ and $R_n x = x - P_n x$ where $x \in [x_n]$ and $\{x_i^*\}_{i=1}^\infty$ are the coefficient functionals of (x_n) . Recall that $\mathcal{K} := \sup_n \|P_n\|$. Following [30] we will denote by Se the Banach space of all sequences of real numbers (c_n) with $\sum_n c_n < \infty$, endowed with its natural norm $\|(c_n)\|_{Se} := \sup_k |\sum_{n=1}^k c_n|$. As usual, c_{00} denotes the vector space of sequences of real numbers which eventually vanish. The *summing basis* is the sequence (e_n) of unit vectors in Se , i.e., $e_n = (\delta_{ni})_{i=1}^\infty$ where δ_{ij} stands for the Kronecker delta.

Definition 2.2. A sequence (x_n) in X dominates another sequence (y_n) if there exists a constant $L > 0$ so that $\|\sum_{n=1}^\infty a_n y_n\| \leq L \|\sum_{n=1}^\infty a_n x_n\|$ for all sequence $(a_n) \in c_{00}$.

When (x_n) and (y_n) are both basic sequences this is equivalent to say that the map $x_n \mapsto y_n$ extends to a linear bounded map between $[x_n]$ and $[y_n]$. One also says that (x_n) and (y_n) are equivalent (or L -equivalent, $L \geq 1$), and one writes $(x_n) \sim_L (y_n)$ if for any $(a_i) \in c_{00}$ it follows that

$$\frac{1}{L} \left\| \sum_{i=1}^\infty a_i x_i \right\| \leq \left\| \sum_{i=1}^\infty a_i y_i \right\| \leq L \left\| \sum_{i=1}^\infty a_i x_i \right\|.$$

The following result will be useful later.

Proposition 2.3. Let (x_n) be a basic sequence in X and $(\alpha_n) \subset (0, 1]$ a non-decreasing sequence of real numbers. Then $(x_n) \sim_{2\mathcal{K}/\alpha_1} (\alpha_n x_n)$ where \mathcal{K} is the basic constant of (x_n) .

Proof. Let $L = 2\mathcal{K}/\alpha_1$. The fact that (x_n) L -dominates $(\alpha_n x_n)$ is not trivial, however, this was essentially proved by Hájek-Johanis [16]. To see this, it suffices to take an equivalent norm on $[x_n]$ so that in the new norm the basis (x_n) is monotone. Indeed, denote by P_I the natural projection over a finite interval $I \subset \mathbb{N}$ and define a new norm on $[x_n]$ by

$$\|x\| = \sup \{ \|P_I x\| : I \subset \mathbb{N}, I \text{ finite interval} \} \quad \text{for } x \in [x_n].$$

Hence $\|\cdot\|$ and $\|\cdot\|$ are equivalent norms on $[x_n]$ with $\max\{\|P_n\|, \|R_n\|\} \leq 1$, $n \in \mathbb{N}$. By [16, Lemma 5-(a)] we have

$$\left\| \sum_{i=1}^{\infty} a_i \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq 2\mathcal{K} \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \quad \text{for all } (a_i) \in c_{00}.$$

To prove the reverse inequality, fix $N \in \mathbb{N}$ and pick any sequence of scalars $(a_i)_{i=1}^N$. Now combining the Abel's partial summation

$$\sum_{n=1}^N a_n \alpha_n x_n = \sum_{n=1}^{N-1} (\alpha_n - \alpha_{n+1}) \sum_{i=1}^n a_i x_i + \alpha_N \sum_{i=1}^N a_i x_i,$$

with the $\|\cdot\|$ -monotonicity of (x_n) , it follows that

$$\left\| \sum_{n=1}^N a_n \alpha_n x_n \right\| \geq \alpha_N \left\| \sum_{i=1}^N a_i x_i \right\| - \sum_{n=1}^{N-1} (\alpha_{n+1} - \alpha_n) \left\| \sum_{i=1}^n a_i x_i \right\| \geq \alpha_1 \left\| \sum_{i=1}^N a_i x_i \right\|$$

which in turn yields

$$2\mathcal{K} \left\| \sum_{n=1}^N a_n \alpha_n x_n \right\| \geq \alpha_1 \left\| \sum_{n=1}^N a_n x_n \right\|.$$

The proof is complete. \square

In the sequel we will rely on the following notions introduced by Rosenthal.

Definition 2.4 ([30]). A seminormalized sequence (x_n) in X is called:

- (i) A wide-(s) sequence if (x_n) is basic and dominates the summing basis.
- (ii) An (s)-sequence if (x_n) is weak-Cauchy and a wide-(s) sequence.

Rosenthal [30] proved that every non-trivial weak-Cauchy sequence in X has either a strongly summing subsequence or a convex block basis which is equivalent to the summing basis. The structure-theoretical approach in our context prompts a different perspective outlined by the following.

Definition 2.5. Let (x_n) be a sequence in X . A sequence (z_n) is called a convex basic sequence of (x_n) if (z_n) is basic and for each $n \in \mathbb{N}$ there exist scalars $\{\lambda_k^{(n)}\}_{k=1}^{\infty}$ in $[0, 1]$ so that $z_n = \sum_{i=1}^{\infty} \lambda_i^{(n)} x_i$ and $\sum_{i=1}^{\infty} \lambda_i^{(n)} = 1$.

Remark. Observe that if a sequence (y_n) contains any subsequence (x_n) equivalent to the ℓ_1 -basis, then every subsequence (z_n) of (x_n) defines a convex basic sequence of (x_n) equivalent to (x_n) .

Now, as we have seen in the introduction, one alternative way of describing weak-compactness in terms of \mathcal{G} -FPP for uniformly Lipschitz maps is by trying to get wide-(s) sequences which dominate all of their subsequences. The next result shows however that spaces with the scalar-plus-compact property do not constitute the most propitious environments for trying to do that.

Proposition 2.6. Let (x_n) be a wide-(s) sequence in X . Assume that (z_n) is a convex basic sequence of (x_n) whose subsequences are dominated by (x_n) . Then $\mathcal{L}([x_n])$ is non-separable.

Proof. We proceed as in [1] obtaining uncountable many pairwise separated bounded linear operators on $[x_n]$. For each increasing sequence (κ_n) in \mathbb{N} define a map $T_{(\kappa_n)}: [x_n] \rightarrow [x_n]$ by $T_{(\kappa_n)}(x) = \sum_{n=1}^{\infty} x_n^*(x) z_{\kappa_n}$. By assumption each $T_{(\kappa_n)} \in \mathcal{L}([x_n])$. Moreover, if \mathcal{K} denotes the basic constant of (z_n) , and $(\kappa_n), (\ell_n)$ are two different increasing sequences in \mathbb{N} then for some $j \in \mathbb{N}$ so that $\kappa_j \neq \ell_j$, we have

$$\begin{aligned} \|T_{(\kappa_n)} - T_{(\ell_n)}\| &\geq \left\| \sum_{n=1}^{\infty} \left(x_n^* \left(\frac{x_j}{\|x_j\|} \right) z_{\kappa_n} - x_n^* \left(\frac{x_j}{\|x_j\|} \right) z_{\ell_n} \right) \right\| \\ &= \frac{1}{\|x_j\|} \|z_{\kappa_j} - z_{\ell_j}\| \geq \frac{\inf_n \|z_n\|}{\mathcal{K} \sup_n \|x_n\|} > 0. \end{aligned}$$

□

We now present the main result of this section.

Theorem 2.7. *Every seminormalized sequence (y_n) in a Banach space X without weakly convergent subsequences has a wide-(s) subsequence (x_n) which admits an equivalent convex basic sequence.*

Proof. If (y_n) has no weak-Cauchy subsequence, (y_n) has an ℓ_1 -subsequence (x_n) by the Rosenthal ℓ_1 -theorem. In this case, because of Remark 2, we get for free the result.

Now assume that (y_n) has a weak-Cauchy subsequence. By Proposition 2.2 of [30], (y_n) has an (s)-subsequence (x_n) . It follows in particular that it is wide-(s).

Let \mathcal{K} be the basic constant of (x_n) and fix any sufficiently small $\varepsilon > 0$. In the sequel we shall build an equivalent convex basic sequence of (x_n) . Let us begin with by choosing for each $n \in \mathbb{N}$ a sequence of positive real numbers $\{\lambda_k^{(n)}\}_{k=1}^{\infty}$ from B_{ℓ_1} , satisfying

- (1) $(1 - \sum_{k=n}^{\infty} \lambda_k^{(n)})_{n \in \mathbb{N}}$ is a non-decreasing sequence in $(0, 1)$.
- (2) $1/2 \leq 1 - \sum_{k=n}^{\infty} \lambda_k^{(n)} \rightarrow 1$ as $n \rightarrow \infty$,
- (3) and, moreover,

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \lambda_k^{(n)} < \frac{\varepsilon}{4\mathcal{K}} \frac{\inf_n \|x_n\|}{\sup_n \|x_n\|}.$$

We now define (z_n) by

$$(1) \quad z_n := \left(1 - \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)} \right) x_n + \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)} x_k, \quad n \in \mathbb{N}.$$

Let us prove that (z_n) is a seminormalized basic sequence. Clearly it is bounded. Since (x_n) is wide-(s) we have that $\inf_n \|z_n\| > 0$. Define an auxiliary sequence $(w_n)_{n \in \mathbb{N}}$ by $w_n = (1 - \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)}) x_n$ and note that (w_n) is seminormalized and basic. Let $\varepsilon_n := \|\sum_{k=n}^{\infty} \lambda_k^{(n)} x_k\|$, $n \in \mathbb{N}$. By comparing (z_n) with (w_n) , we can now directly see that

$$2\mathcal{K} \sum_{n=1}^{\infty} \frac{\|z_n - w_n\|}{\|w_n\|} \leq \frac{4\mathcal{K}}{\inf_n \|x_n\|} \sum_{n=1}^{\infty} \varepsilon_n < 1.$$

So, by the Principle of Small Perturbation (z_n) is basic and hence convex basic.

We are going to prove that (x_n) dominates (z_n) . Let (a_i) be any sequence of scalars in c_{00} and set $\alpha_n = 1 - \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)}$, $n \in \mathbb{N}$. A direct calculation shows that

$$\sum_{i=1}^{\infty} a_i z_i = \sum_{i=1}^{\infty} a_i \alpha_i x_i + \sum_{i=1}^{\infty} a_i \sum_{k=i+1}^{\infty} \lambda_k^{(i+1)} x_k,$$

so that

$$\left\| \sum_{i=1}^{\infty} a_i z_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i \alpha_i x_i \right\| + \sup_n |a_n| \sum_{i=1}^{\infty} \varepsilon_i.$$

On the one hand, we can easily verify that

$$|a_n| \leq \frac{2\mathcal{K}}{\inf_n \|x_n\|} \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \quad \forall n \in \mathbb{N}.$$

On the other hand, by Proposition 2.3 we have

$$\left\| \sum_{i=1}^{\infty} a_i \alpha_i x_i \right\| \leq 2\mathcal{K} \left\| \sum_{i=1}^{\infty} a_i x_i \right\|.$$

It follows therefore that

$$\left\| \sum_{i=1}^{\infty} a_i z_i \right\| \leq \left(2\mathcal{K} + \frac{2\mathcal{K}}{\inf_n \|x_n\|} \sum_{i=1}^{\infty} \varepsilon_i \right) \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq (2\mathcal{K} + \varepsilon) \left\| \sum_{i=1}^{\infty} a_i x_i \right\|.$$

To finish the proof it remains to prove that (z_n) dominates (x_n) . To this end, fix $N \in \mathbb{N}$ and take arbitrary scalars $(a_i)_{i=1}^N$. Note that

$$\left\| \sum_{n=1}^N a_n z_n \right\| \geq \left\| \sum_{n=1}^N a_n \alpha_n x_n \right\| - \sup_n |a_n| \sum_{n=1}^N \left\| \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)} x_k \right\| \geq (1 - \varepsilon) \left\| \sum_{n=1}^N a_n \alpha_n x_n \right\|.$$

The result finally follows from Proposition 2.3. \square

The principal consequence of above result is as follows.

Theorem 2.8. *Let $C \in \mathcal{B}(X)$ be a non weakly compact set. Then C contains a wide-(s) sequence (x_n) with basic constant \mathcal{K} such that:*

- (i) *If $\varepsilon > 0$ is small enough, then there exists a convex basic sequence (z_n) in C such that $(x_n) \sim_{(2\mathcal{K}+\varepsilon)} (z_n)$.*
- (ii) *There exists a bi-Lipschitz affine map $f_{(z_n)}: K \rightarrow K$ which is fixed-point free on $K = \overline{\text{co}}(\{x_n\})$.*

Proof. Since C is non-weakly compact we can choose (y_n) in C with no weakly convergent subsequence. Let (x_n) be the wide-(s) subsequence of (y_n) given by Theorem 2.7, take $\varepsilon > 0$ small enough and define (z_n) as in (1). It is clear that $z_n \in C$ for all n . Notice also that the last inequality above in the previous proof implies $(x_n) \sim_{(2\mathcal{K}+\varepsilon)} (z_n)$.

To complete the proof of theorem it remains to prove the assertion (ii). To do that let $K = \overline{\text{co}}(\{x_n\})$ and note that since (x_n) is wide-(s) one can prove that

$$K = \left\{ \sum_{n=1}^{\infty} t_n x_n : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

We now define a mapping $f_{(z_n)}: K \rightarrow C$ by

$$f_{(z_n)}\left(\sum_{n=1}^{\infty} t_n x_n\right) = \sum_{n=1}^{\infty} t_n z_n.$$

It is clear that $f_{(z_n)}$ is affine and maps K into K . In addition, using that $(x_n) \sim_{(2K+\varepsilon)} (z_n)$ we deduce that $f_{(z_n)}$ is bi-Lipschitz.

We claim that f is fixed point free. Indeed, fix any $x = \sum_{n=1}^{\infty} t_n x_n \in K$ and suppose that $x = f_{(z_n)}(x)$. Note that

$$f_{(z_n)}(x) = t_1\left(\alpha_1 x_1 + \sum_{k=2}^{\infty} \lambda_k^{(2)} x_k\right) + \cdots + t_n\left(\alpha_n x_n + \sum_{k=n+1}^{\infty} t_k^{(n+1)} x_k\right) + \cdots,$$

where $\alpha_n = 1 - \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)}$, $n \in \mathbb{N}$. Thus, $x = f_{(z_n)}(x)$ implies that

$$\begin{cases} t_1 = t_1 \alpha_1 \\ t_2 = t_1 \lambda_2^{(2)} + t_2 \alpha_2 \\ \vdots \\ t_{n+1} = t_1 \lambda_{n+1}^{(2)} + t_2 \lambda_{n+1}^{(3)} + \cdots + t_n \lambda_{n+1}^{(n+1)} + t_{n+1} \alpha_{n+1}, \quad n \geq 2. \end{cases}$$

As $0 < \alpha_1 < 1$ it follows that $t_1 = 0$. Having shown this, assume that for $n \geq 2$ we have proven that $t_1 = t_2 = \cdots = t_n = 0$. Then $t_{n+1} = t_{n+1} \alpha_{n+1}$ and hence since $0 < \alpha_{n+1} < 1$, we have that $t_{n+1} = 0$. Thus each $t_n = 0$, by induction. This contradicts the fact that $\sum_{n=1}^{\infty} t_n = 1$. The proof of theorem is complete. \square

As a corollary to the theorem above, we get the following result that improves the Benavides-Pineda-Pruss theorem (Theorem 1.2).

Theorem 2.9. *Let X be a Banach space and $C \in \mathcal{B}(X)$. Then C is weakly compact if and only if C has the \mathcal{G} -FPP for affine bi-Lipschitz maps.*

3. THE \mathcal{WG} -FPP FOR AFFINE NONEXPANSIVE MAPS

In this section we are interested in describing weak-compactness in terms of the weak-generic fixed point property (\mathcal{WG} -FPP). This new property is a relaxation of the \mathcal{G} -FPP, and asks for fixed points of affine maps that are nonexpansive in a weak sense, that is, when 1-Lipschitzness condition is replaced by the action of weaker topologies.

Definition 3.1. A set $C \in \mathcal{B}(X)$ is said to have the \mathcal{WG} -FPP for affine nonexpansive maps if whenever $K \in \mathcal{B}(C)$ and d is a weaker metric on K , then every affine d -nonexpansive map $f: K \rightarrow K$ has a fixed point.

Next gives an example of the type of theorems we are heading toward.

Theorem 3.2. *Let X be a Banach space. Then $C \in \mathcal{B}(X)$ is weakly compact if and only if it has the \mathcal{WG} -FPP for affine nonexpansive maps.*

Observe that this result contrasts sharply with Lin's theorem, in so far as the denial of the weak-compactness of sets $C \in \mathcal{B}((\ell_1, \|\cdot\|_{\mathcal{L}}))$ can be interpreted in terms of the failure of the \mathcal{WG} -FPP for affine nonexpansive mappings.

In the sequel we present the main results of this section.

We shall assume henceforth, unless stated to the contrary, that X is a Hausdorff locally convex spaces (Hlcs). Let us begin with the following extension of the concept of \mathcal{WG} -FPP.

Definition 3.3. A set $C \in \mathcal{B}(X)$ is said to have the \mathcal{WG} -FPP for affine continuous maps if whenever $K \in \mathcal{B}(C)$ and τ is a weaker Hausdorff topology on K , then every affine τ -continuous mapping $f: K \rightarrow K$ has a fixed point.

Remark. It is worth noting that in above definition τ does not have to be inherited from a linear topology.

There are two important ingredients behind Theorem 3.2. The first one is Mazur's lemma: if (x_n) is weakly convergent sequence in a normed space X , then there is a convex block subsequence of (x_n) which is norm convergent. A convex block subsequence of (x_n) is a sequence (y_k) of the form

$$y_k = \sum_{n \in I_k} \lambda_n x_n$$

where (I_k) is a sequence of finite subsets of \mathbb{N} with $\max(I_k) < \min(I_{k+1})$ and (λ_n) is a sequence of non-negative real numbers so that $\sum_{n \in I_k} \lambda_n = 1$ for all $k \in \mathbb{N}$. A key role in proving this property is played by the norm of X . It is perhaps not surprising that there is a locally convex version of it. However, we have not been able to find any reference to the corresponding extension in the literature. Presuming it to have been missed, we shall try to fill the gap with one embracing a large class of spaces, including several of non-metrizable ones (Lemma 4.2). From now, we will use the following.

Definition 3.4. X is said to have the property (ML) if every weakly null sequence has a convex block subsequence that strongly converges.

Recall that a weaker Hausdorff vector topology σ in a tvs X is said to be compatible with the topology of X if they have the same closed convex sets. The first main result of this section reads as follows.

Theorem 3.5. *Let X be a tvs having property (ML) . Assume that σ is a compatible locally convex topology on X . Then every σ -sequentially compact convex subset of X has the \mathcal{WG} -FPP for affine continuous maps.*

Now, let $\mathfrak{MB}(X)$ denote the subfamily of $\mathcal{B}(X)$ consisting of sets C such that $\overline{\text{aco}}(C)$ is metrizable. The second ingredient is a modified version of Bessaga-Pełczyński's principle (Lemma 4.7). It will be useful for detecting unconditional structures locally spread inside non-weak compact sets, which allows to prove the second main result in this section.

Theorem 3.6. *Let X be a Hlcs. Assume that $C \in \mathfrak{MB}(X)$ is complete and non-weakly compact. Then there exist $K \in \mathcal{B}(C)$, a weaker metric d on K and an affine d -nonexpansive mapping $f: K \rightarrow K$ that is fixed-point free.*

As a consequence, we obtain the following generalization of Theorem 3.2:

Corollary 3.7. *Let X be a Hlcs. Assume that bounded subsets of X are metrizable. Then complete sets $C \in \mathcal{B}(X)$ are weakly compact if and only if they have the \mathcal{WG} -FPP for affine nonexpansive mappings.*

Theorem 3.6 admits the following variation, whose proof can be performed using the classical idea of embedding Hlcs into a product of Banach spaces and applying similar arguments as in [4].

Theorem 3.8. *Let X be a Hlcs and $C \in \mathcal{B}(X)$ a complete, non-weakly compact set. Then there exist $K \in \mathcal{B}(C)$ and an affine continuous mapping $f: K \rightarrow K$ that is fixed-point free.*

Motivated by [3, Proposition 2.5] the last main result of this section highlights the role of weak-approximation of fixed points ($wAFP$) in obtaining FPP . Given a convex subset C of X , let $\mathfrak{A}\mathfrak{C}(C, X)$ (resp. $\mathfrak{A}\mathfrak{C}(C, C)$) denote the family of all affine continuous mappings from C into X (resp. itself). Let w denote the weak-topology of X and set

$$\mathfrak{A}\mathfrak{C}(C, X; wAFP) = \left\{ f \in \mathfrak{A}\mathfrak{C}(C, X) : \exists \text{ net } (u_\alpha) \subset C \text{ s.t. } u_\alpha - f(u_\alpha) \xrightarrow{w} 0 \right\}.$$

The last main result of this section is as follows.

Theorem 3.9. *Let X be a tvs with a weaker Hausdorff compatible locally convex topology σ . Assume that $C \subset X$ is a σ -compact convex set and $f \in \mathfrak{A}\mathfrak{C}(C, X)$. Then f has a fixed point if and only if $f \in \mathfrak{A}\mathfrak{C}(C, X; wAFP)$.*

As an immediate consequence of these results we obtain two corollaries. The first of them was recently proved by Jachymski in [17].

Corollary 3.10. *Let X be a reflexive Banach space and $C \in \mathcal{B}(X)$. Assume that $f \in \mathfrak{A}\mathfrak{C}(C, X)$. Then f has a fixed point if and only if $\inf_{x \in C} \|x - f(x)\| = 0$.*

Corollary 3.11. *Let $C \in \mathcal{B}(X)$ be a complete subset of a Hlcs X . Then C is weakly compact if and only if it has the \mathcal{G} -FPP for maps in $\mathfrak{A}\mathfrak{C}(C, C)$.*

Remark. It is noteworthy mentioning that Corollary 3.11 has already been stated in Floret's Book [11, p. 92] according to the following phrase:

*"In a locally convex space a complete, bounded convex subset A is weakly compact
if
and only if for every closed convex subset $B \subset A$ each affine continuous map
 $B \rightarrow B$
has a fixed point."*

K. Floret [11] has attributed this statement to D. P. and D. V. Milman ([24, 25, 26]), and he has also indicated that a proof could be found in James' paper [18]. Nevertheless, we could not find a written proof of it anywhere. And, as far as we can ascertain from [24, 25, 26], their result (cf. [26, Theorem 2.5]) only yields the following characterization:

Theorem 3.12 (D. P. and D. V. Milman). *A Banach space X is reflexive if and only if every closed convex bounded subset of X has the FPP for affine continuous maps.*

Theorem 3.9 also allows to extending Milman & Milman's characterization of reflexive spaces to the class $\mathfrak{A}\mathfrak{C}(C, X)$ as follows.

Theorem 3.13. *A Banach space X is reflexive if and only if every closed convex bounded subset of X has the FPP for the class $\mathfrak{A}\mathfrak{C}(C, X; AFP)$.*

4. A FEW TECHNICALITIES FOR $\mathcal{W}\mathcal{G}$ -FPP

The goal of this section is to gather all the main tools that we need to establish the $\mathcal{W}\mathcal{G}$ -FPP. The results provided herein are consequences of refinements of methods and approaches available in the literature. Throughout this section, unless otherwise stated, X will denote a Hlcs with topology \mathcal{T} . We begin with an improvement of a result of Drewnowski [9, Theorem 4].

Lemma 4.1. *Let M be a bounded convex subset of X and A its closed absolutely convex hull. Assume that the point 0 in $M - M$ has a countable base of neighborhoods. Then the linear span \mathbb{E} of A admits a norm $\|\cdot\|$ satisfying:*

- (i) *the $\|\cdot\|$ -topology coincides with \mathcal{T} on A ,*
- (ii) *$\|\cdot\|$ -Cauchy sequences in A are \mathcal{T} -Cauchy, and*
- (iii) *weak convergent sequences in A are $\sigma(X, X^*)$ -convergent.*

Proof. Without loss of generality we can assume that $0 \in M$. By assumption there exists a sequence $(U_n)_{n \in \mathbb{N}}$ of absolutely convex \mathcal{T} -neighborhoods of 0 in X with

$$U_{n+1} + U_{n+1} \subset U_n \quad \text{for all } n \in \mathbb{N},$$

and such that the sets $(M - M) \cap U_n$ form a base at 0 in $(M - M, \mathcal{T}|_{M-M})$. In fact, examining the proof of [9, Theorem 4] we see that even the sets $(\overline{M - M}) \cap U_n$ form a base for $\mathcal{T}|_{\overline{M - M}}$ at 0 , where the bar means the \mathcal{T} -closure in X .

The locally convex topology τ on X induced by $(U_n)_{n \in \mathbb{N}}$ is therefore weaker than \mathcal{T} and is such that $\tau|_{\mathbb{E}}$ is Hausdorff. Pick a sequence of seminorms (p_n) on \mathbb{E} defining $\tau|_{\mathbb{E}}$. Since $\tau|_{\mathbb{E}}$ is Lindelöf and A is bounded we can represent A in the form $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is τ -open in A , bounded in X and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. After a suitable scaling, we can assume that $p_n(x) \leq 1$ for all $x \in A_n$. The required norm is defined as

$$\|x\| := \sum_{n=1}^{\infty} 2^{-n} p_n(x) \quad \text{for } x \in \mathbb{E}.$$

The easy argument used in [9, p. 326] also yields (i). Now we will prove (ii). Assume that (z_n) is $\|\cdot\|$ -Cauchy in A and fix any \mathcal{T} -neighborhood U of 0 in X . Observe that

$$(A - A)/2 \subset \overline{M - M}.$$

Thus using the fact that $\{(\overline{M - M}) \cap U_n\}_{n \in \mathbb{N}}$ form a base for $\mathcal{T}|_{\overline{M - M}}$ at 0 , we can find integers $i_1, \dots, i_m \geq 1$ and real numbers $\varepsilon_1, \dots, \varepsilon_m > 0$ such that

$$V := \bigcap_{s=1}^m \{x \in \overline{M - M} : p_{i_s}(x) < \varepsilon_s\} \subset U.$$

For $0 < \varepsilon < \min(2^{-i_1}\varepsilon_1, \dots, 2^{-i_m}\varepsilon_m)$, choose $n_\varepsilon \in \mathbb{N}$ so that $\|z_\kappa - z_\ell\| < 2\varepsilon$ for all $\kappa, \ell > n_\varepsilon$. It follows that $p_{i_s}(z_\kappa - z_\ell) < 2\varepsilon_s$ for all $s = 1, \dots, m$, which in turn implies $z_\kappa - z_\ell \in 2V \subset U$ for all $\kappa, \ell > n_\varepsilon$. As U was arbitrary, this yields (ii).

To conclude the proof, assume that $z_k \xrightarrow{w(\mathbb{E})} z$ in A , where $w(\mathbb{E})$ denotes the induced weak-topology of $(\mathbb{E}, \|\cdot\|)$. Let $\varphi \in X^*$ be fixed. Note that the same argument used in (ii) shows that $\|\cdot\|$ -convergence in A implies \mathcal{T} -convergence. Thus $\varphi|_A$ is $\|\cdot\|$ -continuous. By a result of Grothendieck [14, p. 75, Exercise 1(b)] $\varphi|_A$ is continuous with respect to $w(\mathbb{E})$. For completeness' sake, we indicate an alternative proof of this fact using an approximation result of Roelcke ([29, Theorem 4]): as φ

is linear and $\|\cdot\|$ -continuous on A , given any $\varepsilon > 0$ there exists a $\|\cdot\|$ -continuous linear functional ϕ_ε on \mathbb{E} such that

$$|\varphi(x) - \phi_\varepsilon(x)| < \varepsilon \quad \text{for } x \in A.$$

Hence fixed $\varepsilon > 0$, by the triangle inequality we get

$$\limsup_k |\varphi(z_k) - \varphi(z)| \leq 2\varepsilon.$$

This finishes the proof. \square

Remark. Note that the hypothesis on $M - M$ is automatically satisfied when $A = \overline{\text{aco}}(M)$ is metrizable.

As consequence we obtain a locally convex version of Mazur's lemma.

Lemma 4.2. *Assume that (x_n) is a weakly null sequence in X such that $\overline{\text{aco}}\{x_n : n \in \mathbb{N}\}$ is metrizable. Then there exists a convex block subsequence of (x_n) which is strongly null.*

Proof. Let $M = \overline{\text{aco}}(\{x_n : n \in \mathbb{N}\})$, $A = \overline{\text{aco}}(M)$ and let $\mathbb{E} = \text{span}(A)$. By Lemma 4.1 there exists a norm $\|\cdot\|$ on \mathbb{E} whose topology coincides with \mathcal{T} on A , as a subset of X . Using again [29, Theorem 4], it follows that $x_n \xrightarrow{w(\mathbb{E})} 0$ in $(\mathbb{E}, \|\cdot\|)$. By Mazur's lemma there is a convex block subsequence (y_n) of (x_n) so that $\|y_n\| \rightarrow 0$. So, $y_n \xrightarrow{\mathcal{T}} 0$ in X and the proof is over. \square

Corollary 4.3. *If bounded subsets of X are metrizable, then X has property (ML) .*

Remark. We point out that the class of spaces whose bounded sets are metrizable is large, as has been shown in a recent paper of Ruess [31].

The following is a slight variation of [2, Definition 3.1].

Definition 4.4. A bounded sequence (x_n) in X is said to be σ -equivalent to ℓ_1^0 if the linear span \mathbb{E} of its closed convex hull M admits a Hausdorff locally convex topology σ which is weaker than \mathcal{T} on M and is such that the linear map $T_0 : (c_{00}, \|\cdot\|_{\ell_1}) \rightarrow (\mathbb{E}, \sigma)$ given by

$$T_0\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\infty} a_i x_i \quad \text{for } (a_i) \in c_{00},$$

is an isomorphism. We also say that (x_n) is a σ -equivalent ℓ_1^0 -sequence.

Part of the motivation behind this notion is due to the following results.

Proposition 4.5. *Let (x_n) be a bounded sequence in X such that $M = \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$ is metrizable. Then $\text{span}(M)$ admits a weaker metrizable locally convex topology σ that coincides with \mathcal{T} on M and, moreover, (x_n) either has a σ -equivalent ℓ_1^0 -subsequence or a subsequence which is weak-Cauchy with respect to the weak-topology induced by σ .*

Proof. Note that M is metrizable, by assumption, and separable. Therefore the result follows directly from [9, Theorem 2] and the fact that metrizable locally convex spaces have Rosenthal's property (cf. [2, Theorem 3.5] or more generally [31, Theorem 2.1]). \square

Proposition 4.6. *Let $C \in \mathcal{B}(X)$ be a sequentially complete set. Assume that C contains a sequence σ -equivalent to ℓ_1^0 . Then C fails the \mathcal{G} -FPP for affine uniformly Lipschitz maps.*

Proof. Let (x_n) be a sequence in C σ -equivalent to ℓ_1^0 . Let $M = \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. By [2, Proposition 3.2] there exist a constant $\eta > 0$ and a σ -continuous seminorm μ on $\mathbb{E} = \text{span}(M)$ such that

$$(2) \quad \eta \sum_{i=1}^{\infty} |a_i| \leq \mu \left(\sum_{i=1}^{\infty} a_i x_i \right)$$

for all sequence of scalars (a_i) in c_{00} . Consider now the convex set

$$K = \left\{ \sum_{i=1}^{\infty} t_i x_i : \text{each } t_i \geq 0 \text{ and } \sum_{i=1}^{\infty} t_i = 1 \right\}.$$

It is easy to check that $K = M$. Note also that μ is continuous on M . Moreover, if we define a mapping $f : K \rightarrow K$ by

$$f \left(\sum_{i=1}^{\infty} t_i x_i \right) = \sum_{i=1}^{\infty} t_i x_{i+1} \quad \text{for } \sum_{i=1}^{\infty} t_i x_i \in K,$$

then f is fixed-point free and (uniformly Lipschitz), because $\sigma|_M \leq \mathcal{T}|_M$ and for any continuous seminorm ρ on X , (2) implies

$$\rho(f^p(x) - f^p(y)) \leq \frac{2 \sup_n \rho(x_n)}{\eta} \mu(x - y)$$

for every $x, y \in K$ and $p \in \mathbb{N}$. □

Let now $(X, \|\cdot\|)$ be a normed space and $0 < \lambda \leq 1$. Recall ([15]) that a subset G of X^* is called a λ -norming set for X if

$$\sup_{g \in S(G)} |g(x)| \geq \lambda \|x\| \quad \text{for every } x \in X,$$

where $S(G)$ denotes the set of all unit normed vectors of G . The last result of this section is the following.

Lemma 4.7. *Let X be a normed space, $G \subset X^*$ a λ -norming set and (x_n) a seminormalized sequence of vectors in X satisfying the following conditions:*

- (i) $g(x_n) \rightarrow 0$ for all $g \in G$.
- (ii) (x_n) does not possess norm-Cauchy subsequences.

Then there exist a basic subsequence (e_k) of (x_n) and a constant $\eta > 0$ such that

$$(3) \quad \eta \sum_{i=1}^4 |t_i| \leq \left\| \sum_{i=1}^4 t_i e_{\kappa_i} \right\|$$

for every natural numbers $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4$ and all choice of scalars $(t_i)_{i=1}^4$.

Proof. Define an equivalent norm $\|\cdot\|$ on X by

$$\|x\| = \sup_{g \in S(G)} |g(x)|, \quad x \in X.$$

Let (ε_n) be a sequence in $(0, 1)$ with $\prod_{n=1}^{\infty} (1 + \varepsilon_n) < \infty$ and set $n_1 := 1$. Pełczyński's method [28] yields an increasing sequence $n_1 < n_2 < \dots$ such that

$$\left\| \sum_{i=1}^{\kappa} t_i x_{n_i} \right\| \leq \frac{1}{\lambda} \prod_{s=\kappa}^{\ell-1} (1 + \varepsilon_s) \left\| \sum_{i=1}^{\ell} t_i x_{n_i} \right\|$$

for all $1 \leq \kappa < \ell \leq k$ and all choice of scalars $(t_i)_{i=1}^{k+1}$. Thus (x_{n_i}) is basic.

Note that this property is invariant under taking subsequences of (x_{n_i}) . Now condition (ii) yields a constant $\delta > 0$ so that for every finite dimensional subspace F of X there exists $z \in \{x_{n_i} : i \in \mathbb{N}\}$ so that $\text{dist}(z, F) \geq \delta$. So, we can construct by induction a subsequence (e_k) of (x_{n_i}) such that $\text{dist}(e_{k+1}, \text{span}\{e_1, \dots, e_k\}) \geq \delta$ for any $k \in \mathbb{N}$. The final procedure is now the same as in [3, Lemma 4.3]. We only outline the main steps. Let $M = \sup_n \|x_n\|$ and set $c_i = \delta^{i-1}/2^{2i+1} M^{i-1}$ for $i = 1, \dots, 4$, so we can find the greatest index i_0 so that $|t_i| \geq c_i \sum_{i=1}^4 |t_i|$. Then

$$\left\| \sum_{i=1}^4 t_i x_{\kappa_i} \right\| \geq \left(\delta c_{i_0} - M \sum_{i=i_0+1}^4 c_i \right) \sum_{i=1}^4 |t_i| \geq \frac{1}{2} c_{i_0} \delta \sum_{i=1}^4 |\alpha_i| \geq \frac{1}{32} \frac{\delta^4}{M^3} \sum_{i=1}^4 |\alpha_i|.$$

□

5. PROOF OF THEOREM 3.5

Let K be a closed convex subset of C , τ a weaker Hausdorff topology on K and $f: K \rightarrow K$ an affine τ -continuous mapping. Let us prove that f has a fixed point. By making a translation of K , we can assume that $0 \in K$. Since σ is compatible on X , K is bounded. Then we may construct a sequence (x_n) in K such that $x_n - f(x_n) \rightarrow 0$. Indeed, following [3] we let $y_1 = 0$ and $y_{k+1} = f(y_k)$ for all $k \in \mathbb{N}$. Then the desired sequence is

$$x_n := \frac{y_1 + \dots + y_n}{n} \quad \text{for } n \in \mathbb{N}.$$

From the σ -sequential compactness of C we may pass to a subsequence of (x_n) , again denoted by (x_n) , such that $x_n \xrightarrow{\sigma} u$ for some $u \in \overline{K}^{\sigma}$. Thus $u \in K$ by the compatibility property. Further, since $X^* = (X, \sigma)^*$ we have $x_n \xrightarrow{w} u$ in X , where w denotes the weak topology of X . Since X satisfies property (ML) , there exist a sequence (I_k) of finite subsets of \mathbb{N} with $\max(I_k) < \min(I_{k+1})$ and a sequence (a_n) of non-negative real numbers with $\sum_{n \in I_k} a_n = 1$ for all $k \in \mathbb{N}$ such that $u_k := \sum_{n \in I_k} a_n x_n \rightarrow u$. Now using that τ is weaker than the original topology of X , we get $u_k \xrightarrow{\tau} u$ in K . It follows that $f(u_k) \xrightarrow{\tau} f(u)$. On the other hand, an easy calculation shows that

$$f(u_k) = u_k + \sum_{n \in I_k} a_n \frac{y_{n+1}}{n} \quad \text{for all } k \in \mathbb{N}.$$

This ensures that $f(u_k) \rightarrow u$ in X . Consequently, $f(u_k) \xrightarrow{\tau} u$. Bearing in mind that τ is Hausdorff, we conclude that $f(u) = u$. □

6. PROOF OF THEOREM 3.6

Let \mathcal{T} denote the topology of X . Pick a sequence (y_n) in C with no weak convergent subsequence and let $M = \overline{\text{co}}(\{y_n : n \in \mathbb{N}\})$, where the bar means the \mathcal{T} -closure in X . We may and will assume that $0 \in M$, so $M \subset M - M$. Set $A = \overline{\text{aco}}(M)$ and consider the space $\mathbb{E} = \text{span}(A)$. As observed in [9, p. 328], $A \subset \overline{M} - \overline{M}$ and

so $\mathbb{E} = \overline{\text{span}(M - M)}$. Also, since $\overline{M - M} \subset 2A$ and $C \in \mathfrak{MB}(X)$, the point 0 in M has a countable base of neighborhoods. Let $\|\cdot\|$ be the norm on \mathbb{E} given by Lemma 4.1. Using Proposition 4.6, we can suppose that (y_n) has no subsequence $\|\cdot\|$ -equivalent to ℓ_1^0 . We can therefore choose a $w(\mathbb{E})$ -Cauchy subsequence (x_n) of (y_n) .

Let us check the conditions of Lemma 4.7 for $(\mathbb{E}, \|\cdot\|)$. That (ii) holds follows easily from the fact that $\|\cdot\|$ -Cauchy sequences in M are \mathcal{T} -Cauchy. To verify (i) we define a functional $\Psi \in \tilde{\mathbb{E}}^{**}$ as $\tilde{\mathbb{E}}^* \ni \phi \mapsto \lim_n \phi(J_{\mathbb{E}}(x_n))$, where $\tilde{\mathbb{E}}$ is the completion of \mathbb{E} given by the canonical isometric embedding $J_{\mathbb{E}} : \mathbb{E} \rightarrow \tilde{\mathbb{E}}^{**}$. Then Ψ does not belong to $\tilde{\mathbb{E}}$. Indeed, if this assertion is false, let $\tilde{x} \in \tilde{\mathbb{E}}$ so that $J_{\mathbb{E}}(x_n) \rightarrow \tilde{x}$ in $\tilde{\mathbb{E}}$. So $\tilde{x} \in \overline{J_{\mathbb{E}}(M)}$, by Mazur's lemma. Since $\|\cdot\|$ -Cauchy sequences in M are \mathcal{T} -Cauchy and M is complete, we deduce that $\tilde{x} = J_{\mathbb{E}}(x)$ for some $x \in M$. This implies that $x_n \rightarrow x$ in $(\mathbb{E}, \|\cdot\|)$ and thus $x_n \rightarrow x$ in the weak topology of X , by property (iii) of Lemma 4.1. This contradiction shows that $\Psi \in \tilde{\mathbb{E}}^{**} \setminus \tilde{\mathbb{E}}$. By [15, Lemma I.1.11], $G := \Psi^\perp$ is a λ -norming set for $\tilde{\mathbb{E}}$. It clear that $g(J_{\mathbb{E}}(x_n)) \rightarrow 0$ for all $g \in G$. Let (ε_n) be a sequence of positive real numbers so that $\prod_{s=1}^\infty (1 + \varepsilon_s) < \infty$. By Lemma 4.7, there exist a constant $\eta > 0$ and a basic subsequence (e_k) of (x_n) such that

$$\eta \sum_{i=1}^4 |\alpha_i| \leq \left\| \sum_{i=1}^4 \alpha_i e_{\kappa_i} \right\|,$$

for all sequence of scalars $(\alpha_i)_{i=1}^4$ and indices $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4$. Since $0 \in M$ and (x_n) does not have weak convergent subsequences we can, possibly after passing to a subsequence of (e_k) and taking suitable scalings, suppose that there exists a functional $\phi \in (\mathbb{E}, \|\cdot\|)^*$ so that $\phi(e_k) = 1$ for all $k \in \mathbb{N}$. Consider the convex set

$$K = \left\{ \sum_{k=1}^\infty t_k e_k : \text{each } t_k \geq 0 \text{ and } \sum_{k=1}^\infty t_k = 1 \right\}.$$

It can be easily shown using ϕ that K is \mathcal{T} -closed in C . Let d be a metric on K given by

$$d(x, y) = \frac{\eta}{2\mathcal{K}} \sup_{N \in \mathbb{N}} \sum_{n=N}^{N+3} |t_n - s_n| \quad \text{for } x = \sum_k t_k e_k, y = \sum_k s_k e_k \in K.$$

Clearly the d -topology is weaker than $\mathcal{T}|_K$. Define now a mapping $f : K \rightarrow K$ as follows: for $x = \sum_{k=1}^\infty t_k e_k \in K$, let

$$f(x) = \sum_{k=1}^\infty t_k e_{k+1}.$$

A direct calculation shows that f is fixed-point free and d -nonexpansive. \square

7. PROOF OF THEOREM 3.9

We need an intersection principle.

Lemma 7.1 ([19, 10]). *Let K be a subset of a Hausdorff tvs X . Assume that $F : K \rightarrow 2^X$ is a set-valued map with the following properties:*

- (i) $F(x)$ is closed for all $x \in K$.
- (ii) $F(x_0)$ is compact for some point $x_0 \in K$.

(iii) For every finite family $\{x_1, \dots, x_\ell\} \subset K$ one has

$$(4) \quad \text{co}\{x_1, \dots, x_\ell\} \subset \bigcup_{i=1}^{\ell} F(x_i).$$

Then the following intersection property holds

$$(5) \quad \bigcap_{x \in K} F(x) \neq \emptyset.$$

7.1. Proof of Theorem 3.9. The "only if" part is trivial. To prove the "if" part, assume that $(u_\alpha) \subset C$ is a weak-approximate fixed point net for f . It means that

$$(6) \quad u_\alpha - f(u_\alpha) \xrightarrow{w} 0$$

with the convergence holding in the weak-topology of X .

Let $X_\sigma = (X, \sigma)$ and note that since σ is Hausdorff, X_σ^* separates points of X . For $\varphi \in X_\sigma^*$, set

$$A_\varphi := \{x \in C : |\varphi(x - f(x))| = 0\}.$$

In order to show that f has a fixed point, it is enough to show that

$$\bigcap_{\varphi \in X_\sigma^*} A_\varphi \neq \emptyset.$$

The starting point for proving this is the observation that all of the sets A_φ are σ -closed. This in turn follows from the facts that σ is compatible on X and both φ and f are affine continuous functions. Hence all we need to prove is that the family $\{A_\varphi : \varphi \in X_\sigma^*\}$ has the finite intersection property for σ . Fix functionals $\varphi_1, \dots, \varphi_m \in X_\sigma^*$ and, for the sake of simplicity, define

$$\|x\| = \sum_{i=1}^m |\varphi_i(x)| \quad \text{for } x \in X.$$

Observe that $\|\cdot\|$ is a σ -continuous seminorm on X . Passing to a subnet, if necessary, we may assume that (u_α) σ -converges to some point u of the σ -compact set C . Next, consider the set

$$K := \{x \in C : \|x - u\| \leq 1\}.$$

We then define a set-valued map F which assigns to each $x \in K$ a set $F(x)$ as given below

$$F(x) = \{y \in K : \|y - f(y)\| \leq \|x - f(x)\|\}.$$

Using again that σ is compatible on X we see that F is a well-defined set-valued map with σ -closed convex values. Let us check property (iii) of Lemma 7.1. Suppose by way of contradiction that it does not happen. This means that for some family $\{x_1, \dots, x_\ell\} \subset K$, there exists a convex combination $y = \sum_{i=1}^{\ell} \lambda_i x_i \in \text{co}\{x_1, \dots, x_\ell\}$ such that

$$(7) \quad \|x_i - f(x_i)\| < \|y - f(y)\| \quad \text{for all } i = 1, \dots, \ell.$$

Note that we have implicitly used the property that K is closed under convex combination of its elements. It turns out that

$$\|y - f(y)\| \leq \sum_{i=1}^{\ell} \lambda_i \|x_i - f(x_i)\| \leq \max_{1 \leq i \leq \ell} \|x_i - f(x_i)\|$$

contradicting (7). Therefore, by Lemma 7.1, we can find $\vartheta \in \bigcap_{x \in K} F(x)$. Using now the convergence $u_\alpha \xrightarrow{\sigma} u$, find an index α_1 so that $u_\alpha \in K$ for all $\alpha_1 < \alpha$. In particular it follows that $\vartheta \in F(u_\alpha)$ for all $\alpha_1 < \alpha$. This in turn yields

$$\|\vartheta - f(\vartheta)\| \leq \|u_\alpha - f(u_\alpha)\| \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \alpha_1 < \alpha$$

which from (6) and the weaker property of σ implies $\vartheta \in \bigcap_{i=1}^m A_{\varphi_i}$. The proof is over. \square

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